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# *NEW INVESTIGATION OF THE LAW OF ERRORS OF OBSERVATION.*

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1. ERRORS of observation being freed of constant and regular errors can be only subject to the laws of probability, and we may conceive an error $\varDelta$ to be the result of two opposing influences; one tending to make the observation greater and the other smaller than the true result. If the increasing influence is the greatest a positive error is the result, and vice versa. Now conceive these influences to be made up from $2\Omega = \infty$ equal element errors $\pm i$, some positive and some negative and let $(\Omega + v)$ times $+i$ and $(\Omega - v)$ times $-i$ occur simultaneously to produce an error $\varDelta$ then we have

(1) $$(\Omega + v)i - (\Omega - v)i = 2vi = \varDelta.$$

To produce an error $\varDelta + d\varDelta$ we must suppose one more $+i$ and one less $-i$ to preserve the constant total number $2\Omega$, that is

(2) $$(\Omega + v + 1)i - (\Omega - v - 1)i = 2(v + 1)i = \varDelta + d\varDelta.$$

We have then

(3) $$i = \tfrac{1}{2}d\varDelta,$$

(4) $$vd\varDelta = \varDelta.$$

We have here two events of equal probability, viz., $+i$ and $-i$ and their simple probability is $= \frac{1}{2}$. The terms of the development

(5) $$\left(\frac{1}{2} + \frac{1}{2}\right)^{2\Omega} = 1$$

will give then the probability of any number of $+i$ and the remaining number of $-i$ to occur simultaneously. The middle term, which is the largest, gives the probability of $+\Omega i$ and $-\Omega i$ occurring simultaneously or what is the same thing that of the error 0. If we denote this by $\varphi_0$ and

so on the following terms on both sides by increasing integer subscripts, then $\varphi_v$ givis the probability of $(\Omega + v)i$ and $-(\Omega - v)i$ or by (1) that of the error $\varDelta$. We have then

(6) $$\varphi_0 = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + 1}{\Omega}$$

$$= \text{probability of error } 0 = \Omega i - \Omega i,$$

$$\varphi_1 = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + 2}{\Omega - 1}$$

$$= \text{probability of error } d\varDelta = (\Omega + 1)i - (\Omega - 1)i,$$

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(7) $$\varphi_v = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + v + 1}{\Omega - v}$$

$$= \text{probability of error } \varDelta = (\Omega + v)i - (\Omega - v)i,$$

(8) $$\varphi_{v+1} = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \frac{\Omega + v + 2}{\Omega - v - 1}$$

$$= \text{probability of error } \varDelta + d\varDelta = (\Omega + v + 1)i - (\Omega - v - 1)i,$$

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$$\varphi_{\Omega-1} = 2^{-2\Omega} \cdot \frac{2\Omega}{1} = \text{probability of error } (\Omega - 1)d\varDelta = (2\Omega - 1)i - i,$$

(9) $$\varphi_\Omega = 2^{-2\Omega} = \text{probability of error } \Omega d\varDelta = 2\Omega i - 0 . i = \infty.$$

These probabilities evidently form a continuous function. Denoting the general term $\varphi_v$ by $\varphi$ then $\varphi_{v+1} = \varphi + d\varphi$, and dividing (8) by (7) we have

$$\frac{\text{probability of } \varDelta + d\varDelta}{\text{probability of } \varDelta} = \frac{\varphi_{v+1}}{\varphi_v} = \frac{\varphi + d\varphi}{\varphi} = \frac{\Omega - v}{\Omega + v + 1}.$$

$$\therefore \qquad \frac{d\varphi}{\varphi} = -\frac{2v + 1}{\Omega + v + 1} = -\frac{2v d\varDelta + d\varDelta}{\Omega d\varDelta + v d\varDelta + d\varDelta}$$

(10) $$= -\frac{2\varDelta + d\varDelta}{\Omega d\varDelta + \varDelta + d\varDelta} \text{ by (4),}$$

and at the limit, since $\Omega d\varDelta = \infty$ by (9),

(11) $$\frac{d\varphi}{\varphi} = -\frac{2\varDelta}{\Omega d\varDelta} = -\frac{2\varDelta d\varDelta}{\Omega d\varDelta^2}.$$

$\Omega d\varDelta^2$ is evidently a finite constant. Since $\Omega d\varDelta^2 = 4\Omega i^2 = 2 \times 2\Omega i^2$, it is double the sum of the squares of the element errors $\pm\, i$, and placing this quantity, viz.,

(12) $$2\Omega i^2 = \tfrac{1}{2}\Omega d\varDelta^2 = \varepsilon^2$$ we have

(13) $$\frac{d\varphi}{\varphi} = -\frac{\varDelta d\varDelta}{\varepsilon^2}.$$

Integrating we obtain $\log \varphi = \log C - \varDelta^2 \div 2\varepsilon^2$. If $\varDelta = 0$ then $\log \varphi_0 = \log C$, hence

$$\varphi = \varphi_0 e^{-\frac{\varDelta^2}{2\varepsilon^2}}. \tag{14}$$

We can determine $\varphi_0$ from (6) for we have

$$\varphi_0 = 2^{-2\Omega} \cdot \frac{2\Omega}{1} \cdot \frac{2\Omega - 1}{2} \cdots \cdot \frac{\Omega + 1}{\Omega} \qquad (\Omega = \infty)$$

$$= 2^{-2\Omega} \cdot \frac{1 \cdot 2 \cdot 3 \cdots \cdot 2\Omega}{1^2 \cdot 2^2 \cdot 3^2 \cdots \cdot \Omega^2} \qquad (\Omega = \infty)$$

$$= 2^{-\Omega} \cdot \frac{1 \cdot 3 \cdot 5 \cdots \cdot (2\Omega - 1)}{1 \cdot 2 \cdot 3 \cdots \cdot \Omega} \qquad (\Omega = \infty)$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots \cdot (2\Omega - 1)}{2 \cdot 4 \cdot 6 \cdots \cdot 2\Omega}. \qquad (\Omega = \infty) \tag{15}$$

By Wallis' theorem

$$\frac{\pi}{2} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots \cdot (2n - 2)^2 2n}{1 \cdot 3^2 \cdot 5^2 \cdots \cdot (2n - 1)^2}; \qquad (n = \infty)$$

$$\therefore \quad (n\pi)^{1/2} = \frac{2 \cdot 4 \cdot 6 \cdots \cdot 2n}{1 \cdot 3 \cdot 5 \cdots \cdot (2n - 1)}; \qquad (n = \infty)$$

$$\therefore \varphi_0 = (\Omega\pi)^{-1/2}. \qquad (\Omega = \infty) \tag{16}$$

This remarkable result proves that the probability to commit no error at all is an absolute constant.

We have by (12)

$$\Omega = \frac{2\varepsilon^2}{d\varDelta^2}; \tag{17}$$

therefore also

$$\varphi_0 = \frac{d\varDelta}{\varepsilon\sqrt{(2\pi)}} \tag{18}$$

and

$$\varphi = \frac{d\varDelta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\varDelta^2}{2\varepsilon^2}}. \tag{19}$$

Let $[\varphi]_a^b$ denote the probability of an error to lie between $a$ and $b$, then

$$\Big[\varphi\Big]_a^b = \int_a^b \frac{d\varDelta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\varDelta^2}{2\varepsilon^2}}. \tag{20}$$

The sum of all the probabilities being certainty $= 1$, we must have

$$\Big[\varphi\Big]_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \frac{d\varDelta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\varDelta^2}{2\varepsilon^2}} = 1; \tag{21}$$

and this is the case since by Laplace's integral

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2} = \sqrt{\pi}. \tag{22}$$

2. Supposing now an ideal case in which all the possible errors, viz., 0, $d\varDelta$, $2d\varDelta$, .... $\varDelta$, $\varDelta + d\varDelta$, .... $(\varOmega - 1)d\varDelta$, $\varOmega d\varDelta$ occur positive and negative exactly in proportion to their probabilities as given by (19). Let $m$ be the total number of errors then we have

$$m\varphi = \frac{m d\varDelta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\triangle^2}{2\varepsilon^2}}$$

= number of times the error $\varDelta$ should occur and

$$m\varphi\varDelta^2 = \frac{m d\varDelta}{\varepsilon\sqrt{(2\pi)}} \varDelta^2 e^{-\frac{\triangle^2}{2\varepsilon^2}}$$

= sum of the squares of the errors $\varDelta$. Following the usual Gaussian notation for sums of similarly formed quantities according to which, for inst.,

$$a_1 + a_2 + \dots a_m = [a],$$
$$a_1^2 + a_2^2 + \dots a_m^2 = [a^2],$$
$$a_1 b_1 + a_2 b_2 + \dots a_m b_m = [ab], \quad \text{etc.},$$

we have the sum of the squares of all the errors in our ideal case

$$\begin{aligned}
[\varDelta^2] &= m\int_{-\infty}^{+\infty} \frac{d\varDelta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\triangle^2}{2\varepsilon^2}} \\
&= \frac{m\varepsilon}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} d\varDelta . -e^{-\frac{\triangle^2}{2\varepsilon^2}} \\
&= -\frac{m\varepsilon}{\sqrt{(2\pi)}}\left[\varDelta e^{-\frac{\triangle^2}{2\varepsilon^2}}\right]_{-\infty}^{+\infty} + \frac{m\varepsilon}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} d\varDelta e^{-\frac{\triangle^2}{2\varepsilon^2}} \\
&= m\varepsilon^2.
\end{aligned}$$

$$(23) \qquad \therefore \quad \varepsilon = \sqrt{\left(\frac{[\varDelta^2]}{m}\right)}.$$

There can be no objection to use this formula for a finite number of errors provided we admit that in the long run the law of frequency of errors which we have assumed in this ideal case to be strictly fulfilled, will hold also in experience the more perfectly the greater the number of observations.

As we shall see, the method which we are going to explain consists mainly in this: to determine the unknown quantities in such a way that the system of resulting errors $\varDelta_1, \varDelta_2, \dots \varDelta_m$ resembles as much as possible the above ideal case, and this must bring us the nearer to truth the more regular the law of error has been followed.

The constant $\varepsilon$ found by (23) is called the mean error of $\varDelta_1, \varDelta_2, \dots \varDelta_m$ and denoting it by $\varepsilon_\triangle$ we have

$$(23') \qquad \varepsilon_\triangle = \sqrt{\left(\frac{\varDelta_1^2 + \varDelta_2^2 + \dots \varDelta_m^2}{m}\right)} = \sqrt{\left(\frac{[\varDelta^2]}{m}\right)},$$

and this quantity is to be distinguished from the mean error $\varepsilon$ of the observations which slightly exceeds $\varepsilon_{\triangle}$ since it includes the uncertainty of the unknown quantities themselves. It is plain from all this that $\varepsilon$ the mean error of observation characterizes a class of observations, and this with regard to precision is plain from the following: If $\varepsilon'$ is the mean error of another system we have the probability of an error $\Delta'$ by (14)

$$\varphi' = \varphi_0 e^{-\frac{\triangle'^2}{2\varepsilon'^2}}.$$

If an error $\Delta$ in the first system is to have the same probability as an error $\Delta'$ in the second then must

$$\frac{\Delta}{\varepsilon\sqrt{2}} = \frac{\Delta'}{\varepsilon'\sqrt{2}}.$$

If then $\varepsilon' > \varepsilon$ then also $\Delta' > \Delta$, that is, greater errors in the second system have the same probability as smller ones in the first.

Gauss puts

(24) $$h = \frac{1}{\varepsilon\sqrt{2}}$$

and calls $h$ the measure of precision, which is the greater the lesser the class of observations.

3. The most usual method to compare the precision of different systems is by means of the probable error. This is a quantity which stands in the middle of the series of errors arranged according to their magnitude. We have then the probability to commit an error less than the probable error equal to that to commit an error greater, or, denoting it by $r$, we have regarding (20) and (21),

(25) $$\int_0^r \frac{d\Delta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\triangle^2}{2\varepsilon^2}} = \int_r^\infty \frac{d\Delta}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\triangle^2}{2\varepsilon^2}} = \frac{1}{4}.$$

Place

(26) $$\Delta \div \varepsilon\sqrt{2} = h\Delta = \theta,$$

then (25) becomes

(27) $$\frac{1}{\sqrt{\pi}}\int_0^{r\div\varepsilon\sqrt{2}} d\theta\, e^{-\theta^2} = \frac{1}{\sqrt{\pi}}\int_0^{\rho} d\theta\, e^{-\theta^2} = \frac{1}{4},$$

where we have put

(28) $$r \div \varepsilon\sqrt{2} = \rho.$$

If we solve (27) for $\rho$, which is most readily done by means of tables of the definite integral

$$\int_0^\theta d\theta\, e^{\theta^2},$$

we find

(29) $$\rho = 0.47694,$$
hence by (28)
(30) $$r = \varepsilon\rho\sqrt{2} = 0.6745\varepsilon.$$

This formula gives the relation betweed mean and probable error. We have then the following relation between mean error, probable error and precision

(31) $$h = \frac{1}{\varepsilon\sqrt{2}} = \frac{\rho}{r},$$

and the probability of an error $\Delta$ is

(32) $$\varphi = \frac{d\triangle}{\varepsilon\sqrt{(2\pi)}} e^{-\frac{\triangle^2}{2\varepsilon^2}} = \frac{hd\triangle}{\sqrt{\pi}} e^{-h^2\triangle^2} = \frac{\rho d\triangle}{r\sqrt{\pi}} e^{-\frac{\rho^2}{r^2}\triangle^2}.$$

4. The probable error is highly significant in its relation to a determined quantity. It is then an even wager that the error of that determination exceeds the probable error as that it is smaller. It is usually placed with a double sign after the qnantity. Thus the equation

$$x_1 = a_1 \pm r_1$$

signifies that the most probable value of the unknown quantity $x_1$ has been found $a_1$ with such an uncertainty that the actual error of $a_1$ is with the same probability greater as it is less than $r_1$. The probability of any value to be $= x_1$ is therefore

(33) $$\varphi(x_1) = \frac{\rho dx_1}{r_1\sqrt{\pi}} e^{-\frac{\rho^2}{r_1^2}(x_1-a_1)^2}.$$

Similarly we have if $x_2 = a_2 \pm r_2$

(33′) $$\varphi(x_2) = \frac{\rho dx_2}{r_2\sqrt{\pi}} e^{-\frac{\rho^2}{r_2^2}(x_2-a_2)^2}.$$

If now

(34) $$X = \alpha_1 x_1 + \alpha_2 x_2$$

and we require the most probable value and probability of $x$ we cannot assume the first to be $\alpha_1 a_1 + \alpha_2 a_2$, at least not without proof. As this will be furnished as soon as we know the probability of $X$ we shall determine this probability.

We evidently pass through all imaginable values for $X$ if we combine in (34) any value of $x_1$ with any value of $x_2$; hence the probability of any special value of $X$ is the compound probability

$$\frac{\rho^2 dx_1 dx_2}{r_1 r_2 \pi} e^{-\frac{\rho^2}{r_1^2}(x_1-a_1)^2 - \frac{\rho^2}{r_2^2}(x_2-a_2)^2}$$

after $x_1$ has passed from $+\infty$ to $-\infty$, or

(35) $$\varphi(X) = \frac{\rho^2 dx_2}{r_1 r_2 \pi} \int_{-\infty}^{+\infty} dx_1\, e^{-\frac{\rho^2}{r_1^2}(x_1-a_1)^2 - \frac{\rho^2}{r_2^2}(x_2-a_2)^2}.$$

By (34) we have

(36) $$x_2 = \frac{X - a_1 x_1}{a_2},$$

(37) $$dx_2 = \frac{dX}{a_2},$$

since $x_2$ must be independent of $x_1$ after the $x_1$ integrations. Introducing these values into (35) we have

(38) $$\varphi(X) = \frac{\rho^2 dX}{a_2 r_1 r_2 \pi}\int_{-\infty}^{+\infty} dx_1\, e^{-\frac{\rho^2}{r_1^2}(x_1-\alpha_1)^2 - \frac{\rho^2}{a_2^2 r_2^2}(X - a_1 x_1 - a_2\alpha_2)^2}.$$

The exponent may be put in the form $-Ax_1^2 + 2Bx_1 - C$, hence

$$\varphi(X) = \frac{\rho^2 dX}{a_2 r_1 r_2 \pi}\int_{-\infty}^{+\infty} dx_1\, e^{-A(x_1-\frac{B}{A})^2 + \frac{B^2}{A} - C}$$

$$= \frac{\rho^2 dX}{a_2 r_1 r_2 \pi} e^{-(C-\frac{B^2}{A})}\int_{-\infty}^{+\infty} dx_1\, e^{-A(x_1-\frac{B}{A})^2}$$

(39) $$= \frac{\rho^2 dX}{a_2 r_1 r_2 \pi} e^{-(C-\frac{B^2}{A})^{-\infty}} \cdot \sqrt{\frac{\pi}{A}}. \quad \text{[by (21).]}$$

But by comparison

$$A = \frac{\rho^2}{r_1^2} + \frac{\rho^2 a_1^2}{r_2^2 a_2^2} = \frac{\rho^2}{r_1^2 r_2^2 a_2^2}\Big[a_1^2 r_1^2 + a_2^2 r_2^2\Big],$$

$$B = \frac{\rho^2}{r_1^2}\alpha_1 + \frac{\rho^2}{r_2^2}\cdot\frac{a_1}{a_2^2}\Big(X - a_2\alpha_2\Big) = \frac{\rho^2}{r_1^2 r_2^2 a_2^2}\Big[\alpha_1 a_2^2 r_2^2 + (X - a_2\alpha_2)a_1 r_1^2\Big]$$

$$C = \frac{\rho^2}{r_1^2}\alpha_1^2 + \frac{\rho^2}{r_2^2}\cdot\frac{(X - a_2\alpha_2)^2}{a_2^2} = \frac{\rho^2}{r_1^2 r_2^2 a_2^2}\Big[\alpha_1^2 a_2^2 r_2^2 + (X - a_2\alpha_2)^2 r_1^2\Big]$$

$$\frac{B^2}{A} - C = \frac{\rho^2}{r_1^2 r_2^2 a_2^2}\left[\frac{[\alpha_1 a_2^2 r_2^2 + (X - a_2\alpha_2)a_1 r_1]^2}{a_1^2 r_1^2 + a_2^2 r_2^2} - \alpha_1^2 a_2^2 r_2^2 - (X - a_2\alpha_2)^2 r_1^2\right]$$

$$= -\frac{\rho^2}{a_1^2 r_1^2 + a_2^2 r_2^2}\Big[X - a_1\alpha_1 - a_2\alpha_2\Big]^2.$$

With these values (39) becomes

(40) $$\varphi(X) = \frac{\rho\, dX}{\sqrt{(a_1^2 r_1^2 + a_2^2 r_2^2)\pi}} e^{-\frac{\rho^2}{a_1^2 r_1^2 + a_2^2 r_2^2}(X - a_1\alpha_1 - a_2\alpha_2)^2}.$$

The most probable value of $X$ is therefore

(41) $$X_0 = a_1\alpha_1 + a_2\alpha_2,$$

and its probable error

(42) $$R = \sqrt{(a_1^2 r_1^2 + a_2^2 r_2^2},$$

or the complete value

(43) $$X = a_1\alpha_1 + a_2\alpha_2 \pm \sqrt{(a_1^2 r_1^2 + a_2^2 r_2^2)}.$$

If more generally

$$(34') \qquad X = a_1 x_1 + a_2 x_2 + \ldots . a_m x_m = [a\,x],$$

where $\quad x_1 = a_1 \pm r_1,\ x_2 = a_2 \pm r_2, \ldots . x_m = a_m \pm r_m,$
then we have by composition

$$(40') \qquad \varphi(X) = \frac{\rho dX}{\sqrt{([a^2 r^2]\pi)}}\, e^{-\frac{\rho^2}{[a^2 r^2]}(X-[a\,a])^2},$$

$$(43') \qquad X = [a\,a] \pm \sqrt{([a^2 r^2])}.$$

If

$$(34'') \qquad X = f(x_1, x_2, \ldots . x_m)$$

the above integration cannot be effected but an approximate solution can be given in that case which is the nearer perfect the smaller the probable errors $r_1, r_2, \ldots r_m$. We have by Taylor's theorem, neglecting higher powers of increments $\triangle x_1,\ \triangle x_2, \ldots . \triangle x_m$

$$(34''') \quad X = f(a_1, a_2, \ldots a_m) + f'(a_1)\triangle x_1 + f'(a_2)\triangle x_2 + \ldots f'(a_m)\triangle x_m.$$

Within the range of $\triangle x_1, \triangle x_2, \ldots \ldots \triangle x_m$, for which this form is exact enough, $X$ is of the form (34′). In the integration however these increments have to pass from $+\infty$ to $-\infty$. If the probable errors are small this will make no sensible difference since the integral

$$\int_{\triangle}^{\infty} \frac{\rho d\triangle}{r\sqrt{\pi}}\, e^{-\frac{\rho^2}{r^2}\triangle^2}$$

approaches 0 the more rapidly the smaller $r$. This circumstance admits to a certain extent the treating of $X$ as a linear function of $\triangle x_1,\ \triangle x_2, \ldots \triangle x_m$ and we have

$$(40'') \qquad \varphi(X) = \frac{\rho dX}{\sqrt{([f'(a)^2 r^2]\pi)}}\, e^{-\frac{\rho^2}{[f'(a)^2 r^2]}[X - f'(a_1, a_2, \ldots)]^2}$$

$$(43'') \qquad X = f(a_1, a_2, \ldots . a_m) \pm \sqrt{([f'(a)^2 r^2])}.$$

(*To be continued.*)

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Note by S. W. Salmon.—In the note on Differential Calculus (p. 14), I wrote $\left(\frac{y-u'}{u-u'}\right)_{x=x'} = 1$. This needs to be proved. If the rate of motion of the point $B$ is increasing, just before $x = x'$, $\left(\frac{y-u'}{u-u'}\right)$ is greater than 1, and just after $x = x'$, it is less than 1; therefore when $x = x'$, $\left(\frac{y-u'}{u-u'}\right) = 1$. If $B$'s rate is decreasing, it may be proved in a similar manner that $\left(\frac{y-u'}{u-u'}\right)_{x=x'} = 1$.